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Factorization, reduction and embedding in integrable cellular automata

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Abstract

Factorized dynamics in soliton cellular automata with quantum group symmetry is identified with a motion of particles and anti-particles exhibiting pair creation and annihilation. An embedding scheme is presented showing that the $D_n^{(1)}$ -automaton contains, as certain subsectors, the box–ball systems and all the other automata associated with the crystal bases of non-exceptional affine Lie algebras. The results extend the earlier ones to higher representations by a certain reduction and to a wider class of boundary conditions.

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1. Introduction

In [HKT1], a class of one-dimensional (1D) cellular automata with quantum group symmetry was introduced. They are associated with the crystal bases [KKM] of non-exceptional affine Lie algebras $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. Since then, (i) configurations that behave as solitons are extracted [HKT1], (ii) scattering of solitons is determined in terms of combinatorial R by utilizing the quantum group symmetry [HKOTY], (iii) time evolution under a sufficiently large carrier is factorized into a product of Weyl group operators [HKT2], (iv) the resulting factorized dynamics is described by an elementary algorithm using particles and anti-particles [HKT3], (v) tropical R for geometric crystals is formulated [KOTY1], which yields the evolution equation of the automata under ultradiscretization [TTMS] and (vi) the tropical R is bilinearized in terms of tau functions of the DKP hierarchy [KOTY2], allocating the automata in classical integrable systems.

Through the developments these automata have now been understood at a level comparable with the well-known box–ball systems [TS, T, TM] corresponding to $A_n^{(1)}$ [HHIKTT, FOY]. The factorization of the time evolution in this case was known as the decomposition into a finer process where one only moves the balls with a fixed colour.

The states of our automata have the form (1), where a specifies the boundary condition and $l = (l_i)_{i \in \mathbb{Z}}$ denotes the ‘capacities of the aligned boxes’ in the terminology of the box–ball systems and ‘magnitude of local spins’ in the terminology of solvable lattice models.

Throughout the paper the automata with $(\forall l_i = 1)_{i \in \mathbb{Z}}$ and general $(l_i)_{i \in \mathbb{Z}}$ will be called basic and inhomogeneous, respectively.

The goal of this paper is to provide the full generalization of the above result (iv) which was limited to the basic automata with the boundary condition $a = 1$. Our approach involves three essential ingredients: *factorization* (section 2), *reduction* (section 3) and *embedding* (section 4). Factorization of the time evolution into the Weyl group operators had been proved for any inhomogeneous automata associated with $A_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$ in [HKT2].

In sections 2 and 3 we concentrate on the $D_n^{(1)}$ case and the former is devoted to the quotation of the result in [HKT2]. Section 3 contains our main theorem 7, which claims that under any boundary condition a , the factorized dynamics of the inhomogeneous $D_n^{(1)}$ -automaton admits a reduction to the basic one except for inserting a certain ‘reshuffling operator’ Q . See examples 4 and 5. It integrates the crystal basis theory on the time evolution into the simple diagrams (8), which represent a motion of particles and anti-particles that undergo pair creation and annihilation. Until this point we achieve double simplifications of the inhomogeneous $D_n^{(1)}$ -automaton; first by the factorization with respect to ‘colour’ of the particles and second by the reduction to the basic case up to reshuffling. They make the time evolution a handy enough calculation and suit programming due to the locality of the diagrams (8).

In section 4 we explain the embedding, by which the $D_n^{(1)}$ -automaton can be regarded as a ‘master automaton’ that contains, as certain subsectors, all the other inhomogeneous automata associated with $A_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$. In a sense this is the third simplification that unites the number of results obtained previously. Our embedding scheme is a synthesis of the preceding ones in [HKOTY] and [KOTY2].

In the box–ball system associated with $A_{n-1}^{(1)}$, all the colours are equivalent under \mathbb{Z}_n symmetry. Such a symmetry is absent in the relevant crystal graph for $D_n^{(1)}$ in figure 1 in appendix A. This motivates us to study the automaton under a general boundary condition specified by any point a in the crystal graph. Since time evolutions under any boundary conditions are connected by a similarity transformation (see definition 1), qualitative features of the automata remain the same. However, solitons show up in various guises depending on a . Section 6 contains the list of such configurations under the general boundary condition.

We defer the technical details to section 5 and appendix B, where a proof of theorem 7 is outlined. The key is to identify the crystal theoretic Weyl group operator S_i [K] with the particle motion operator K_j via a certain gauge transformation as in proposition 11 and lemma 18. We note that for the box–ball systems, a reduction to the basic case was also known in [F].

2. Factorization in inhomogeneous $D_n^{(1)}$ -automaton

For a positive integer k , let B_k be the crystal for k -fold symmetric tensor representation of $D_n^{(1)}$ ($n \geq 4$) [KKM]. Each element of B_k is written as $\vec{x} = (x_1, \dots, x_n, x_{-n}, \dots, x_{-1})$, where x_i are non-negative integers that add up to k and satisfy the condition $x_n x_{-n} = 0$. Let $J = \{\pm 1, \dots, \pm n\}$. For $a \in J$ we let $\vec{a} \in B_k$ denote the unique element characterized by $x_a = k$. (We suppress the dependence on k .) Let $l = (l_i)_{i \in \mathbb{Z}}$ be an array of positive integers. Given $a \in J$ and $l = (l_i)_{i \in \mathbb{Z}}$, recall the inhomogeneous $D_n^{(1)}$ -automaton [HKT2] having the capacity l and the prescribed boundary condition a . It is a discrete dynamical system over the set

$$W[a]_l = \{(\dots, \vec{b}_i, \vec{b}_{i+1}, \dots) \in \dots \times B_{l_i} \times B_{l_{i+1}} \times \dots \mid \vec{b}_i = \vec{a} \quad \text{for } |i| \gg 1\} \quad (1)$$

where the product \times of sets actually means the tensor product of the crystals B_{l_i} .

Let S_0, \dots, S_n be the Weyl group operators [K] acting on $\prod_{a \in J} W[a]_l$. They satisfy the Coxeter relations, which include in particular $S_i^2 = \text{id}$, $S_0 S_1 = S_1 S_0$ and $S_n S_{n-1} = S_{n-1} S_n$. We introduce the operator σ^Δ that acts on $\mathbf{b} = (\dots, \vec{b}_i, \vec{b}_{i+1}, \dots) \in \prod_{a \in J} W[a]_l$ as $\sigma^\Delta(\mathbf{b}) = (\dots, \sigma(\vec{b}_i), \sigma(\vec{b}_{i+1}), \dots)$, where $\sigma(\vec{b}_i)$ is obtained from $\vec{b}_i = (x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in B_i$ by the simultaneous interchanges $x_1 \leftrightarrow x_{-1}$ and $x_n \leftrightarrow x_{-n}$. It induces the involutive automorphism $\sigma^\Delta S_0 \sigma^\Delta = S_1$, $\sigma^\Delta S_{n-1} \sigma^\Delta = S_n$ and $\sigma^\Delta S_i \sigma^\Delta = S_i$ for $i \neq 0, n$.

In terms of these operators the time evolution $\mathcal{T}_a : W[a]_l \rightarrow W[a]_l$ of our $D_n^{(1)}$ -automaton is defined as follows.

Definition 1.

$$\begin{aligned} \mathcal{T}_1 &= \sigma^\Delta S_0 S_2 S_3 \cdots S_{n-2} S_n S_{n-1} \cdots S_2 S_0 \\ \mathcal{T}_a &= S_{a-1}^{-1} \mathcal{T}_{a-1} S_{a-1} \quad (\text{for } 2 \leq a \leq n) \\ &= \begin{cases} \sigma^\Delta S_a S_{a+1} \cdots S_{n-2} S_n S_{n-1} \cdots S_2 S_0 S_1 \cdots S_{a-1} & 2 \leq a \leq n-1 \\ \sigma^\Delta S_{n-1} S_{n-2} \cdots S_2 S_0 S_1 \cdots S_{n-1} & a = n \end{cases} \\ \mathcal{T}_{-n} &= S_n^{-1} \mathcal{T}_{n-1} S_n = \sigma^\Delta S_n S_{n-2} \cdots S_2 S_0 S_1 \cdots S_{n-2} S_n \\ \mathcal{T}_{-a} &= S_a^{-1} \mathcal{T}_{-a-1} S_a \quad (\text{for } 1 \leq a \leq n-1) \\ &= \begin{cases} \sigma^\Delta S_{a-1} S_{a-2} \cdots S_2 S_0 S_1 \cdots S_{n-2} S_n S_{n-1} \cdots S_a & 2 \leq a \leq n-1 \\ \sigma^\Delta S_1 S_2 S_3 \cdots S_{n-2} S_n S_{n-1} \cdots S_2 S_1 & a = 1. \end{cases} \end{aligned}$$

These operators can be extended to $\prod_{a \in J} W[a]_l$, where $\mathcal{T}_a = (\mathcal{T}_{-a})^{-1}$ is valid for any $1 \leq a \leq n$. See also (20) and (21) for a property of the Weyl group elements.

Example 2. For $n = 4$, \mathcal{T}_a read

$$\begin{aligned} \mathcal{T}_1 &= \sigma^\Delta S_0 S_2 S_4 S_3 S_2 S_0 \\ \mathcal{T}_2 &= \sigma^\Delta S_2 S_4 S_3 S_2 S_0 S_1 \\ \mathcal{T}_3 &= \sigma^\Delta S_4 S_3 S_2 S_0 S_1 S_2 \\ \mathcal{T}_4 &= \sigma^\Delta S_3 S_2 S_0 S_1 S_2 S_3 \\ \mathcal{T}_{-4} &= \sigma^\Delta S_4 S_2 S_0 S_1 S_2 S_4 \\ \mathcal{T}_{-3} &= \sigma^\Delta S_2 S_0 S_1 S_2 S_4 S_3 \\ \mathcal{T}_{-2} &= \sigma^\Delta S_0 S_1 S_2 S_4 S_3 S_2 \\ \mathcal{T}_{-1} &= \sigma^\Delta S_1 S_2 S_4 S_3 S_2 S_1. \end{aligned}$$

The automaton was originally introduced in [HKT1] for $\forall l_i = 1$ and $a = 1$ case, where the time evolution \mathcal{T}_1 was defined by using the combinatorial R . It was extended to general inhomogeneity l and boundary condition a in [HKT2], where it was proved that the \mathcal{T}_a is factorized as above into the Weyl group operators. Due to the boundary condition, it can be shown that all the S_i act actually as $S_i = e_i^\infty$. The expressions in definition 1 do not depend on l explicitly. The basic $D_n^{(1)}$ -automaton corresponding to the situation $\forall l_i = 1$ will play a fundamental role in the following.

3. Reduction to basic $D_n^{(1)}$ -automaton

For the basic $D_n^{(1)}$ -automaton, the time evolution \mathcal{T}_1 in definition 1 was described as a motion of particles and anti-particles that undergo pair creation and annihilation [HKT3]. Here we generalize the result to the inhomogeneous $D_n^{(1)}$ -automaton for any \mathcal{T}_a . It turns out that the

inhomogeneous automaton admits a reduction to the basic one except for the insertion of a certain operator Q which will be introduced around (6).

First we prepare some notation for the basic automaton. We identify B_1 with J as a set via the correspondence $\vec{a} \in B_1 \leftrightarrow a \in J$. Then states of the basic $D_n^{(1)}$ -automaton are infinite arrays of the letters from J . The totality of them will be denoted by

$$\tilde{W}[a] = \{(\dots, b_i, b_{i+1}, \dots) \in \dots \times B_1 \times B_1 \times \dots \mid b_i = a \text{ for } |i| \gg 1\}. \quad (2)$$

In the following, we introduce the operators P, Q, L_j and K_j . Except for P , they are actually dependent on the choice a of the boundary condition. (In appendix B we will write them as $Q^{(a)}$ and $K_j^{(a)}$.) For $\vec{x} = (x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in B_k$, let $P(\vec{x})$ be an array of k letters arranged as

$$P(\vec{x}) = \overbrace{-1 \dots -1}^{x_{-1}} \dots \overbrace{-n \dots -n}^{x_{-n}} \overbrace{n \dots n}^{x_n} \dots \overbrace{1 \dots 1}^{x_1}. \quad (3)$$

For $\vec{b} = (\dots, \vec{b}_i, \vec{b}_{i+1}, \dots) \in W[a]_l$, let $P(\vec{b})$ be an array of infinite letters and walls given by

$$P(\vec{b}) = \dots |P(\vec{b}_i)|P(\vec{b}_{i+1})| \dots \quad (4)$$

where each $P(\vec{b}_i)$ for $\vec{b}_i \in B_{l_i}$ is defined by (3). The symbol $|$ denotes the wall inserted between every $P(\vec{b}_i)$ and $P(\vec{b}_{i+1})$.

Let $\tilde{W}[a]_l$ be the same set as $\tilde{W}[a]$ except that walls are inserted at the same positions as in the elements of $P(W[a]_l)$. Namely $\tilde{W}[a]_l$ consists of the elements of the form

$$c = \dots |C_i|C_{i+1}| \dots$$

where $C_i (i \in \mathbb{Z})$, which we call a *cell*, is an array of l_i letters from J . For distant i , i.e., $|i| \gg 1$, the cell C_i consists only of the letter a . On the other hand C_i with not necessarily distant i can assume different ordering from (3), hence $P(W[a]_l) \subset \tilde{W}[a]_l$ holds.

We let $\tilde{W}[a]_l^Q$ denote a subset of $\tilde{W}[a]_l$ in which every cell C_i has the form

$$C_i = \overbrace{-a \dots -a}^{s_i} \dots \overbrace{a \dots a}^{t_i} \quad (5)$$

involving $\pm a$ nowhere else in the cell C_i . We let Q be the operator on $\tilde{W}[a]_l^Q$ that sends each cell C_i in (5) to

$$C'_i = \overbrace{a \dots a}^{t_i} \dots \overbrace{-a \dots -a}^{s_i} \quad (6)$$

and keeps the other letters unchanged. By the definition Q becomes trivial for the basic automaton.

For $a \in J$ and $j \in J \setminus \{a, -a\}$ we introduce a map $L_j : (\mathbb{Z}_{\geq 0}) \times B_1 \rightarrow B_1 \times (\mathbb{Z}_{\geq 0})$ as follows. Let the diagram

$$\begin{array}{ccc} & b & \\ m \downarrow & \downarrow & \downarrow \\ & m' & \\ & b' & \end{array} \quad (7)$$

depict the relation $L_j : (m, b) \mapsto (b', m')$. Then under the assumption $m \in \mathbb{Z}_{\geq 0}$ and $b \in B_1 \setminus \{j, -j, a, -a\}$, the following specify the map L_j completely:

$$\begin{array}{ccccccc} \begin{array}{ccc} -j & & \\ m+1 \downarrow & \downarrow & \downarrow \\ & m & \\ -a & & \end{array} & \begin{array}{ccc} -j & & \\ 0 \downarrow & \downarrow & \downarrow \\ & 0 & \\ -j & & \end{array} & \begin{array}{ccc} -a & & \\ m \downarrow & \downarrow & \downarrow \\ & m+1 & \\ -j & & \end{array} & \begin{array}{ccc} j & & \\ m \downarrow & \downarrow & \downarrow \\ & m+1 & \\ a & & \end{array} & \begin{array}{ccc} a & & \\ m+1 \downarrow & \downarrow & \downarrow \\ & m & \\ j & & \end{array} & \begin{array}{ccc} a & & \\ 0 \downarrow & \downarrow & \downarrow \\ & 0 & \\ a & & \end{array} & \begin{array}{ccc} b & & \\ m \downarrow & \downarrow & \downarrow \\ & m & \\ b & & \end{array} \quad (8) \end{array}$$

For $j \in J \setminus \{a, -a\}$ we construct an operator $K_j : \tilde{W}[a] \rightarrow \tilde{W}[a]$ from a composition of L_j as follows. Given a state of the basic automaton $(\dots, b_i, b_{i+1}, \dots) \in \tilde{W}[a]$, there exists an integer g such that $b_{g'} = a$ for all $g' < g$ owing to the boundary condition (2). Fix any such g . Then the operator K_j maps $(\dots, b_i, b_{i+1}, \dots)$ to $(\dots, c_i, c_{i+1}, \dots)$, where $c_{g'} = b_{g'}$ for all $g' < g$. The remaining c_g, c_{g+1}, \dots are determined by the composition of L_j as

$$\begin{array}{ccccccc}
 & b_g & b_{g+1} & \cdots & & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
 0 & \rightarrow & \rightarrow & \rightarrow & \cdots & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
 & c_g & c_{g+1} & \cdots & & &
 \end{array} \tag{9}$$

It is easy to see that the result is independent of the choice of g . We let K_j act also on $\tilde{W}[a]_l$ in the same way as $\tilde{W}[a]$ by ignoring the walls. For an example, if $a = 3$, the action of K_2 on a state $\dots, -2, 2, -3, 3, 4, -3, -2, 3, 3, \dots \in \tilde{W}[3]$ proceeds as (\dots denotes the array of 3 only)

$$K_2 \left(\begin{array}{cccccccccccc}
 & -2 & 2 & -3 & 3 & 4 & -3 & -2 & 3 & 3 & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 0 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & -2 & 3 & -2 & 2 & 4 & -2 & -3 & 2 & 3 & & & & & & & & &
 \end{array} \right)$$

leading to the output $\dots, -2, 3, -2, 2, 4, -2, -3, 2, 3, \dots \in \tilde{W}[3]$.

Let us present an interpretation as a system of particles and anti-particles with various colours. A state $(\dots, b_i, b_{i+1}, \dots) \in \tilde{W}[a]$ is regarded as a one-dimensional array of boxes. The local state $b_i = b$ with $b \in B_1 \setminus \{a, -a\}$ means that the i th box contains a particle of colour b . Since the set $B_1 \setminus \{a, -a\}$ is invariant under the interchange $b \leftrightarrow -b$, the local state $-b$ may be viewed as a box containing an anti-particle of b and vice versa. We regard the selected local state $a \in B_1$ as an empty box. On the other hand, when acting K_j on $\tilde{W}[a]$, the local state $-a \in B_1$ plays a special role as a bound state of a particle of colour j and its anti-particle $-j$. The boundary condition in (2) allows only finitely many boxes to be non-empty. Under this interpretation, the operator K_j describes the dynamics of right-moving particles of colour j seeking the empty box a or a free partner, i.e., an anti-particle $-j$ not yet paired with the other j 's, to form a bound state $-a$ within a box. To see this, regard the diagram (7) as representing m incident particles with colour j coming from the left and m' particles with the same colour j outgoing to the right. The propagation of the colour j particles to the right thereby induces the change of contents in a box corresponding to the transformation $b \rightarrow b'$. In fact, in the first figure of (8), a colour j particles coming from the left is trapped in a box to form the bound state $-a$ with the anti-particle $-j$ (pair annihilation $(j, -j) \rightarrow -a$). In the second figure, there are no colour j particles coming from the left nor in the box hence nothing happens. In the third figure, a colour j particle is removed off the bound state $-a$ leaving the anti-particle $-j$ (pair creation $-a \rightarrow (j, -j)$). In the fourth figure, a colour j particle is taken away to the right leaving an empty box a . In the fifth figure, a colour j particle stops at the empty box to fill it. In the last two figures, nothing can happen. When K_j acts on $\tilde{W}[a]_l$ instead of $\tilde{W}[a]$, it induces the same motion of particles without any effect from the walls. In contrast with this, the operator Q remembers the walls at $l = (l_i)_{i \in \mathbb{Z}}$ and swaps the positions of the empty boxes a and the bound states $-a$ within each cell.

Definition 3. For $1 \leq a \leq n$ we define the operator T_a on $W[a]_l$ as

$$T_a = P^{-1} \left(\prod_{j=a+1}^n K_j \right) \left(\prod_{j=-n}^{-(a+1)} K_j \right) \left(\prod_{j=-(a-1)}^{-1} K_j \right) Q \left(\prod_{j=1}^{a-1} K_j \right) P \tag{10}$$

and T_{-a} on $W[-a]_l$ as

$$T_{-a} = P^{-1} \left(\prod_{j=-(a-1)}^{-1} K_j \right) Q \left(\prod_{j=1}^{a-1} K_j \right) \left(\prod_{j=a+1}^n K_j \right) \left(\prod_{j=-n}^{-(a+1)} K_j \right) P \quad (11)$$

where $\prod_{j=c}^{c-1} K_j = \text{id}$, and for $c \leq d$ we adopt the convention $\prod_{j=c}^d K_j = K_c K_{c+1} \cdots K_d$.

One should remember that the operators K_j and Q (and L_j) are actually dependent on a as opposed to P . Up to this distinction, the expressions of T_n and T_{-n} look formally identical. See remark 20 for the possible rearrangement of these products.

Example 4. The time evolution operators for inhomogeneous $D_4^{(1)}$ -automaton read as

$$\begin{aligned} T_1 &= P^{-1} K_2 K_3 K_4 K_{-4} K_{-3} K_{-2} Q P \\ T_2 &= P^{-1} K_3 K_4 K_{-4} K_{-3} K_{-1} Q K_1 P \\ T_3 &= P^{-1} K_4 K_{-4} K_{-2} K_{-1} Q K_1 K_2 P \\ T_4 &= P^{-1} K_{-3} K_{-2} K_{-1} Q K_1 K_2 K_3 P \\ T_{-4} &= P^{-1} K_{-3} K_{-2} K_{-1} Q K_1 K_2 K_3 P \\ T_{-3} &= P^{-1} K_{-2} K_{-1} Q K_1 K_2 K_4 K_{-4} P \\ T_{-2} &= P^{-1} K_{-1} Q K_1 K_3 K_4 K_{-4} K_{-3} P \\ T_{-1} &= P^{-1} Q K_2 K_3 K_4 K_{-4} K_{-3} K_{-2} P. \end{aligned}$$

Each operator $K_j (j \in J)$ in definition 3 transforms the cells as

$$\dots \overbrace{a \dots a}^{\alpha} \overbrace{j \dots j}^{\beta} \dots \overbrace{-j \dots -j}^{\gamma} \overbrace{-a \dots -a}^{\delta} \dots \mapsto \dots \overbrace{j \dots j}^{\alpha'} \overbrace{a \dots a}^{\beta'} \dots \overbrace{-a \dots -a}^{\gamma'} \overbrace{-j \dots -j}^{\delta'} \dots \quad (12)$$

for $-n \leq j \leq -1$, and as

$$\dots \overbrace{-j \dots -j}^{\gamma} \overbrace{-a \dots -a}^{\delta} \dots \overbrace{a \dots a}^{\alpha} \overbrace{j \dots j}^{\beta} \dots \mapsto \dots \overbrace{-a \dots -a}^{\gamma'} \overbrace{-j \dots -j}^{\delta'} \dots \overbrace{j \dots j}^{\alpha'} \overbrace{a \dots a}^{\beta'} \dots \quad (13)$$

for $1 \leq j \leq n$, where $\alpha + \beta = \alpha' + \beta'$ and $\gamma + \delta = \gamma' + \delta'$. The letters other than $\pm j$ and $\pm a$ are kept unchanged. In (12) and (13), the ordering of the letters are the same as (3) except for the positions of $\pm a$. See example 14. This fact assures that the right-hand sides of (10) and (11) without the leftmost P^{-1} belong to $P(W[a]_l)$, hence definition 3 makes sense.

Example 5. Consider T_{-4} for $D_4^{(1)}$ given in example 4. The operators K_j and Q therein act as (to save space we write $\bar{4}$ to mean -4 , etc)

$$\begin{aligned} &\dots |\bar{3}431|\bar{1}\bar{2}|\bar{4}2|\bar{4}31|\bar{4}\bar{4}\bar{4}|\dots \\ \xrightarrow{K_3} &\dots |\bar{3}\bar{3}41|\bar{1}\bar{2}|32|3\bar{4}1|3\bar{4}\bar{4}|\dots \\ \xrightarrow{K_2} &\dots |\bar{3}\bar{3}41|\bar{1}\bar{2}|3\bar{4}|321|3\bar{4}\bar{4}|\dots \\ \xrightarrow{K_1} &\dots |\bar{3}\bar{3}\bar{4}\bar{4}|\bar{4}\bar{2}|3\bar{4}|32\bar{4}|31\bar{4}|\dots \\ \xrightarrow{Q} &\dots |\bar{4}\bar{4}\bar{3}\bar{3}|\bar{2}\bar{4}|\bar{4}3|\bar{4}32|\bar{4}31|\dots \\ \xrightarrow{K_{-1}} &\dots |\bar{4}\bar{4}\bar{3}\bar{3}|\bar{2}1|\bar{1}3|\bar{4}32|\bar{4}31|\dots \\ \xrightarrow{K_{-2}} &\dots |\bar{4}\bar{4}\bar{3}\bar{3}|\bar{4}1|\bar{1}3|\bar{2}32|\bar{4}31|\dots \\ \xrightarrow{K_{-3}} &\dots |\bar{4}\bar{4}\bar{4}\bar{4}|\bar{3}1|\bar{1}\bar{4}|\bar{2}32|\bar{4}31|\dots \end{aligned}$$

where \dots stands for the cells that consist of $\bar{4}$ only. Both the initial and the final states belong to $P(W[-4]_l)$ with $l = (\dots, 4, 2, 2, 3, 3, \dots)$.

For $a \neq \pm n$, the time evolution T_a involves the factor $K_n K_{-n}$. At the intermediate state between K_n and K_{-n} , the cells can contain the letters n and $-n$ simultaneously, but a and $-a$ cannot coexist. For the other intermediate states, the situation is opposite, namely, a and $-a$ can coexist but n and $-n$ cannot. The following is such an example. See also example 14.

Example 6. Consider T_1 for $D_4^{(1)}$ given in example 4. The operators K_j and Q therein act as (to save space we write $\bar{4}$ to mean -4 etc)

$$\begin{aligned}
 & \dots |\bar{4}\bar{4}|\bar{3}431|\bar{1}\bar{3}\bar{4}|\bar{2}|\bar{1}\bar{1}|111|\dots \\
 \xrightarrow{Q} & \dots |\bar{4}\bar{4}|\bar{1}\bar{3}43|\bar{3}\bar{4}\bar{1}|\bar{2}|\bar{1}\bar{1}|111|\dots \\
 \xrightarrow{K_{-2}} & \dots |\bar{4}\bar{4}|\bar{1}\bar{3}43|\bar{3}\bar{4}2|1|\bar{2}2|\bar{2}\bar{2}1|\dots \\
 \xrightarrow{K_{-3}} & \dots |\bar{4}\bar{4}|\bar{1}14\bar{1}|\bar{1}42|\bar{3}|\bar{2}2|\bar{2}\bar{2}1|\dots \\
 \xrightarrow{K_{-4}} & \dots |11|\bar{4}\bar{4}44|\bar{4}12|\bar{3}|\bar{2}2|\bar{2}\bar{2}\bar{4}|\dots \\
 \xrightarrow{K_4} & \dots |11|\bar{4}\bar{4}11|\bar{1}42|\bar{3}|\bar{2}2|\bar{2}\bar{2}\bar{4}|\dots \\
 \xrightarrow{K_3} & \dots |11|\bar{4}\bar{4}11|\bar{3}42|\bar{1}|\bar{2}2|\bar{2}\bar{2}\bar{4}|\dots \\
 \xrightarrow{K_2} & \dots |11|\bar{4}\bar{4}11|\bar{3}41|\bar{2}|\bar{1}\bar{1}|\bar{1}\bar{1}\bar{4}|\dots
 \end{aligned}$$

where \dots stands for the cells that consist of 1 only. Both the initial and the final states belong to $P(W[1]_l)$ with $l = (\dots, 2, 4, 3, 1, 2, 3, \dots)$.

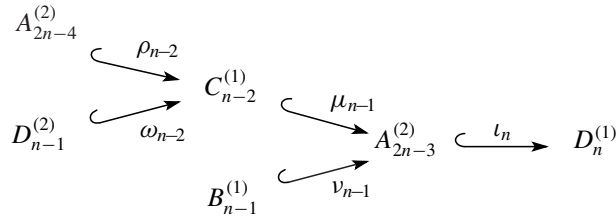
Now we give the main theorem of this paper.

Theorem 7. For any $a \in J, T_a = T_a$ is valid. Namely, the time evolution operator T_a in definition 1 coincides with T_a in definition 3 that admits the particle interpretation.

We include a proof in section 5 and appendix B.

4. Embedding into inhomogeneous $D_n^{(1)}$ -automaton

In this section an automaton means an inhomogeneous automaton. The $D_n^{(1)}$ -automaton ($n \geq 4$) contains, as certain sectors, all the automata associated with the Kang–Kashiwara–Misra (KKM) crystals [KKM] for the other non-exceptional affine Lie algebras $A_{2n-3}^{(2)}, A_{2n-4}^{(2)}, B_{n-1}^{(1)}, C_{n-2}^{(1)}$ and $D_{n-1}^{(2)}$. Here we describe the embedding of these automata along the following scheme:



By embedding an automaton, we precisely mean the following. Take any pair $\psi : X \hookrightarrow X'$ in the above scheme. Then there is an injection $\psi : B_k \rightarrow B'_k$ between the KKM crystals associated with these algebras X and X' , respectively. Moreover, this ψ has an extension

Table 1. Sets $B_k = \{\vec{x} \mid (*)\}$.

Algebra	Array \vec{x}	Condition (*)
$A_{2n-1}^{(2)}$	$(x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n}$	$s(\vec{x}) = k$
$A_{2n}^{(2)}$	$(x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n}$	$s(\vec{x}) \leq k$
$B_n^{(1)}$	$(x_1, \dots, x_n, x_0, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n+1}$	$x_0 \in \{0, 1\}, s(\vec{x}) = k$
$C_n^{(1)}$	$(x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n}$	$s(\vec{x}) \in k - 2\mathbb{Z}_{\geq 0}$
$D_n^{(1)}$	$(x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n}$	$x_n x_{-n} = 0, s(\vec{x}) = k$
$D_{n+1}^{(2)}$	$(x_1, \dots, x_n, x_0, x_{-n}, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^{2n+1}$	$x_0 \in \{0, 1\}, s(\vec{x}) \leq k$

Table 2. Maps $\iota_n, \mu_n, \nu_n, \rho_n, \omega_n$.

ψ	$\psi(\vec{x})$	$\psi(e_i)$
$\iota_n : B_k \rightarrow B'_k$	$(x_1, \dots, x_{n-1}, 0, 0, x_{-n+1}, \dots, x_{-1})$	e_i
$\mu_n : B_k \rightarrow B'_k$	$(x_0, x_1, \dots, x_{n-1}, x_{-n+1}, \dots, x_{-1}, x_0)$	e_{i+1}
$\nu_n : B_k \rightarrow B'_{2k}$	$(2x_1, \dots, 2x_{n-1}, 2x_n + x_0, 2x_{-n} + x_0, 2x_{-n+1}, \dots, 2x_{-1})$	$e_i^{2-\delta_{i,n}}$
$\rho_n : B_k \rightarrow B'_{2k}$	$(2x_1, \dots, 2x_n, 2x_{-n}, \dots, 2x_{-1})$	$e_i^{2-\delta_{i,0}}$
$\omega_n : B_k \rightarrow B'_{2k}$	$(2x_1, \dots, 2x_{n-1}, 2x_n + x_0, 2x_{-n} + x_0, 2x_{-n+1}, \dots, 2x_{-1})$	$e_i^{2-\delta_{i,0}-\delta_{i,n}}$

to Kashiwara operators [K] e_i, f_i in such a way that $\psi(e_i \vec{x}) = \psi(e_i) \psi(\vec{x})$ and $\psi(f_i \vec{x}) = \psi(f_i) \psi(\vec{x})$ for any $\vec{x} \in B_k$ and i . It follows that the combinatorial R for X is embedded into the one for X' as $R = (\psi^{-1} \otimes \psi^{-1})R'(\psi \otimes \psi)$. The map $\psi^\Delta = (\dots \otimes \psi \otimes \psi \otimes \dots)$ embeds the states of the X -automaton into the X' -automaton and relates their time evolutions simply by $T_a^X = (\psi^\Delta)^{-1} T_{a'}^{X'} \psi^\Delta$. The precise correspondence of the boundary conditions a and a' will be described below together with the injections $\iota_n, \mu_n, \nu_n, \rho_n$ and ω_n . The embeddings ρ_n and ω_n for the basic automata have appeared in [HKOTY]. The μ_n and ι_n have been introduced in section 6.1 of [KOTY2] in a tropical setting.

In table 1, the KKM crystal B_k is listed for each algebra as a set. They all have the presentation $B_k = \{\vec{x} = (x_1, \dots, x_{-1}) \in \mathbb{Z}_{\geq 0}^N \mid (*)\}$ with certain N and the condition (*) on \vec{x} . We let $s(\vec{x}) = x_1 + \dots + x_{-1}$ denote the sum of all the components of \vec{x} and list the array \vec{x} and (*). The images of the maps appearing in the embedding scheme are given in table 2. The component x_0 appearing in the image $\mu_n(\vec{x})$ is the non-negative integer specified by $x_0 = (k - \sum_{j=1}^{n-1} (x_j + x_{-j}))/2$. The image $\psi(f_i)$ is obtained from $\psi(e_i)$ by changing the letter e to f .

As an example let us illustrate the embedding ι_n in further detail. This is a significant case in that the $A_{2n-3}^{(2)}$ -automaton is considerably simpler than the $D_n^{(1)}$ -automaton and all the other automata can first be embedded into the former.

In view of $\iota_n : (x_1, \dots, x_{n-1}, x_{-n+1}, \dots, x_{-1}) \mapsto (x_1, \dots, x_{n-1}, 0, 0, x_{-n+1}, \dots, x_{-1})$, the $A_{2n-3}^{(2)}$ -automaton is a restriction of the $D_n^{(1)}$ -automaton to those local states satisfying $x_n = x_{-n} = 0$. The set of states is formally the same as (1) once B_{l_i} is understood as the one for $A_{2n-3}^{(2)}$. The boundary condition is specified by a letter $a \in \{\pm 1, \dots, \pm(n-1)\}$. The time evolutions T_a for these a are still given by the formulae in definition 3 except only that the states they act on are restricted. The T_a with $a \neq \pm n$ contains the operators K_n and K_{-n} only through the combination $K_n K_{-n}$. See example 4 for $n = 4$. In lemma 21 we will show that

the composition $K_n K_{-n}$ under the restriction $x_n = x_{-n} = 0$ coincides with a simple operator K_{-a} defined as follows. (Note that K_{-a} was not introduced in the $D_n^{(1)}$ case.) The K_{-a} is the composition of L_{-a} in the same way as in (9), and the L_{-a} is specified by the diagrams:

$$\begin{array}{cccc}
 \begin{array}{c} -a \\ \downarrow \\ m \rightarrow m+1 \\ \uparrow \\ a \end{array} &
 \begin{array}{c} a \\ \downarrow \\ m+1 \rightarrow m \\ \uparrow \\ -a \end{array} &
 \begin{array}{c} a \\ \downarrow \\ 0 \rightarrow 0 \\ \uparrow \\ a \end{array} &
 \begin{array}{c} b \\ \downarrow \\ m \rightarrow m \\ \uparrow \\ b \end{array}
 \end{array} \tag{14}$$

where $b \in B \setminus \{a, -a\}$ and $m \in \mathbb{Z}_{\geq 0}$. These diagrams represent the motion of the right-moving particles $-a$ seeking the empty box a without pair creation or annihilation. Such a behaviour of colour $-a$ particles is exceptional, in that they still act as bound states of j and $-j$ under the operator K_j with $j \neq -a$.

Example 8. The time evolution operators for $A_5^{(2)}$ -automaton read as

$$\begin{aligned}
 T_1 &= P^{-1} K_2 K_3 K_{-1} K_{-3} K_{-2} Q P \\
 T_2 &= P^{-1} K_3 K_{-2} K_{-3} K_{-1} Q K_1 P \\
 T_3 &= P^{-1} K_{-3} K_{-2} K_{-1} Q K_1 K_2 P \\
 T_{-3} &= P^{-1} K_{-2} K_{-1} Q K_1 K_2 K_3 P \\
 T_{-2} &= P^{-1} K_{-1} Q K_1 K_3 K_2 K_{-3} P \\
 T_{-1} &= P^{-1} Q K_2 K_3 K_1 K_{-3} K_{-2} P.
 \end{aligned}$$

Example 9. Consider T_3 for $A_5^{(2)}$ given in example 8. The operators K_j and Q therein act as (to save space we write $\bar{3}$ to mean -3 , etc)

$$\begin{aligned}
 &\dots |11|\bar{1}\bar{1}\bar{3}1|\bar{1}32|3333|\dots \\
 \xrightarrow{K_2} &\dots |11|\bar{1}\bar{1}\bar{2}1|\bar{1}23|2333|\dots \\
 \xrightarrow{K_1} &\dots |33|\bar{3}\bar{3}\bar{2}3|\bar{3}23|2333|\dots \\
 \xrightarrow{Q} &\dots |33|\bar{3}\bar{2}\bar{3}\bar{3}|\bar{3}2\bar{3}|3332|\dots \\
 \xrightarrow{K_{-1}} &\dots |33|\bar{3}\bar{2}11|\bar{1}21|\bar{1}\bar{1}32|\dots \\
 \xrightarrow{K_{-2}} &\dots |33|\bar{3}311|\bar{1}\bar{3}1|\bar{1}\bar{1}32|\dots \\
 \xrightarrow{K_{-3}} &\dots |33|\bar{3}311|\bar{1}31|\bar{1}\bar{1}\bar{3}2|\dots
 \end{aligned}$$

where \dots stands for the cells that consist of 3 only.

Let us briefly comment on the $A_{n-1}^{(1)}$ case where the associated automaton is known as the (generalized) box-ball system [TS, T, HHIKTT, FOY]. The relevant crystal B_k is the set $\{\vec{x} = (x_1, \dots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid x_1 + \dots + x_n = k\}$ equipped with the functions $\varphi_i(\vec{x}) = x_i$, $\varepsilon_i(\vec{x}) = x_{i+1}$ and the Kashiwara operators $e_i(\vec{x}) = (\dots, x_i + 1, x_{i+1} - 1, \dots)$, $f_i(\vec{x}) = (\dots, x_i - 1, x_{i+1} + 1, \dots)$, where all the indices are considered to be in $\mathbb{Z}/n\mathbb{Z}$. We set $J = \{1, 2, \dots, n\}$ and $\vec{a} = (x_i = k\delta_{i,a}) \in B_k$ for $a \in J$. In particular $\vec{a} \in B_1$ will simply be called a . Under this notation the automaton states are again specified by (1). Due to the Dynkin diagram \mathbb{Z}_n symmetry, there is no loss of generality in restricting ourselves to the case $a = 1$. Then the time evolution is given by the Weyl group operators as $\mathcal{T}_1 = \sigma^\Delta S_2 S_3 \dots S_{n-1} S_0$ [HKT2], where $\sigma^\Delta = \dots \otimes \sigma \otimes \sigma \otimes \dots$ is defined by $\sigma((x_1, x_2, \dots, x_n)) = (x_2, \dots, x_n, x_1)$.

In the same way as section 5 and appendix B, we have proved that $\mathcal{T}_1 = T_1$ holds, where T_1 consists of the particle motion operator as

$$T_1 = P^{-1}K_2K_3 \cdots K_nQP. \tag{15}$$

Here P acts on $W[1]_l$ as (4) with (3) simplified as

$$P(\vec{x}) = \overbrace{n \dots n}^{x_n} \dots \overbrace{2 \dots 2}^{x_2} \overbrace{1 \dots 1}^{x_1}. \tag{16}$$

The operator Q rearranges each cell (16) into $\overbrace{1 \dots 1}^{x_1} \overbrace{n \dots n}^{x_n} \dots \overbrace{2 \dots 2}^{x_2}$. The operator K_j is defined in terms of the vertex diagrams ($2 \leq j \leq n, b \neq 1, j$):

$$\begin{array}{cccc}
 \begin{array}{c} j \\ \downarrow \\ m \rightarrow m+1 \\ \downarrow \\ 1 \end{array} &
 \begin{array}{c} 1 \\ \downarrow \\ m+1 \rightarrow m \\ \downarrow \\ j \end{array} &
 \begin{array}{c} 1 \\ \downarrow \\ 0 \rightarrow 0 \\ \downarrow \\ 1 \end{array} &
 \begin{array}{c} b \\ \downarrow \\ m \rightarrow m \\ \downarrow \\ b \end{array}
 \end{array} \tag{17}$$

These rules coincide with T_1 of the $D_n^{(1)}$ -automaton restricted to the sector without anti-particles, i.e., those local states $(x_1, \dots, x_n, x_{-n}, \dots, x_{-1})$ such that $x_{-n} = \dots = x_{-1} = 0$. In fact, the absence of anti-particles reduces the diagrams (8) to the last four, which yield (17) under the choice $a = 1$. Thus we conclude that the inhomogeneous box-ball system is reduced to the basic case via (15). Moreover, it coincides with a sector of the $D_n^{(1)}$ -automaton without anti-particles. The result equivalent to (15) has also been obtained by purely combinatorial arguments in [F]. The diagrams equivalent to (17) appeared in [HIK] for the basic box-ball system.

5. Weyl group operator S_i as particle motion K_j

Our main theorem $\mathcal{T}_a = T_a$ for $D_n^{(1)}$ is a composition of finer relations that intertwine the Weyl group operators S_i and the particle motion operators K_j . The aim of this section is to present them in proposition 11. Many assertions and formulae provided below for the general case will be recognized most easily by consulting the $n = 4$ case in examples 13, 14 as well as examples 2 and 4.

For $\vec{x} = (x_1, \dots, x_n, x_{-n}, \dots, x_{-1}) \in B_k$, let $\bar{P}(\vec{x})$ be an array of k letters arranged as

$$\bar{P}(\vec{x}) = \overbrace{1 \dots 1}^{x_1} \overbrace{-2 \dots -2}^{x_2} \dots \overbrace{-n \dots -n}^{x_n} \overbrace{n \dots n}^{x_n} \dots \overbrace{2 \dots 2}^{x_2} \overbrace{-1 \dots -1}^{x_1} \tag{18}$$

which is obtained from $P(\vec{x})$ in (3) by interchanging the positions of 1 and -1 . Similarly to (4), we set $\bar{P}(\vec{b}) = \dots |\bar{P}(\vec{b}_i)|\bar{P}(\vec{b}_{i+1})| \dots$ for $\vec{b} = (\dots, \vec{b}_i, \vec{b}_{i+1}, \dots) \in W[a]_l$.

In section 2 we introduced the Weyl group operator S_j acting on $\coprod_{a \in J} W[a]_l$. Here we use two further kinds of Weyl group operators r_j, s_j ($0 \leq j \leq n$). The $r_j : B_1 \rightarrow B_1$ is the one for the crystal B_1 . In the crystal graph depicted in figure 1, if there is a colour j arrow coming into or outgoing from the node p , the image $r_j(p)$ is determined by the relation $r_j(p) \xrightarrow{j} p$ or $r_j(p) \xleftarrow{j} p$, respectively. If there is no such arrow, $r_j(p) = p$. The s_j is the one that acts on $\coprod_{a \in J} W[a]_l$ totally componentwise ignoring the walls. Namely, it transforms the cells as

$$\begin{array}{ll}
 \mathbf{c} = \dots |C_i|C_{i+1}| \dots & C_i = p_1 \dots p_{l_i} \in B_1^{\times l_i} \\
 s_j(\mathbf{c}) = \dots |C'_i|C'_{i+1}| \dots & C'_i = r_j(p_1) \dots r_j(p_{l_i}) \in B_1^{\times l_i}.
 \end{array}$$

Obviously, one has

$$S_i(W[a]_l) \subseteq W[r_i(a)]_l \quad s_i(\tilde{W}[a]_l) \subseteq \tilde{W}[r_i(a)]_l. \tag{19}$$

For $a \in J$ let

$$\mathcal{T}_a = \sigma^\Delta S_{i_{2n-2}} \cdots S_{i_1} \tag{20}$$

be the expression of \mathcal{T}_a given in definition 1. The sequence $i_1, i_2, \dots, i_{2n-2}$ has the property

$$\varepsilon_{i_t}(r_{i_{t-1}} \cdots r_{i_2} r_{i_1}(a)) > 0 \quad 1 \leq t \leq 2n - 2 \tag{21}$$

where ε_i is defined in (37). The operator T_a in definition 3 is expressed as

$$T_a = P^{-1} Q^{\delta_{a,-1}} K'_{j_{2n-2}} \cdots K'_{j_1} P \quad K'_j \in \{K_j, K_j Q\} \tag{22}$$

with the sequence $j_1, j_2, \dots, j_{2n-2} \in J$ specified by

$$j_t = r_{i_1} r_{i_2} \cdots r_{i_{t-1}} r_{i_t} r_{i_{t-1}} \cdots r_{i_2} r_{i_1}(a) \quad 1 \leq t \leq 2n - 2. \tag{23}$$

In view of (19), the operator S_{i_t} in (20) acts among the sets:

$$S_{i_t} : W[r_{i_{t-1}} \cdots r_{i_1}(a)]_l \rightarrow W[r_{i_t} \cdots r_{i_1}(a)]_l \tag{24}$$

for $a \in J$ and $1 \leq t \leq 2n - 2$.

Let $r : B_1 \rightarrow B_1$ be the involutive automorphism $1 \leftrightarrow -1$ and $n \leftrightarrow -n$ keeping the other letters unchanged. It is easy to check that $r_{j_1} \cdots r_{i_{2n-2}} = r$ for all $a \in J$. Consequently $s(= s^{-1}) = s_{i_1} \cdots s_{i_{2n-2}} : \coprod_{a \in J} \tilde{W}[a]_l \rightarrow \coprod_{a \in J} \tilde{W}[a]_l$ is also independent of a , and induces r componentwise.

Lemma 10.

$$\sigma^\Delta \bar{P}^{-1} s = P^{-1} \quad \text{on } P \left(\coprod_{b \in J} W[b]_l \right) \tag{25}$$

$$\sigma^\Delta P^{-1} s = P^{-1} Q \quad \text{on } \bar{P} \left(\coprod_{b \in J} W[b]_l \right) \tag{26}$$

where Q is the one for the $a = -1$ case in (5), (6).

The essential task in proving theorem 7 is to establish

Proposition 11. For any $a \in J$ and $1 \leq t \leq 2n - 2$, the action of the Weyl group operator S_{i_t} in (24) is expressed as follows:

$$S_{i_t} = P^{-1} s_{i_t} \cdots s_{i_1} K_{j_t} s_{i_1} \cdots s_{i_{t-1}} P \quad \text{if } K'_{j_t}, \dots, K'_{j_1} \text{ do not contain } Q \tag{27}$$

$$= \bar{P}^{-1} s_{i_t} \cdots s_{i_1} K_{j_t} Q s_{i_1} \cdots s_{i_{t-1}} P \quad \text{if } K'_{j_t} = K_{j_t} Q \tag{28}$$

$$= \bar{P}^{-1} s_{i_t} \cdots s_{i_1} K_{j_t} s_{i_1} \cdots s_{i_{t-1}} \bar{P} \quad \text{otherwise.} \tag{29}$$

In fact, substituting (27)–(29) into (20) and reducing the result by lemma 10, $s_i^2 = \text{id}$ and $s_{i_{2n-2}} \cdots s_{i_1} = s$, one gets (22). Note that $a = -1$ is the unique case where (28) and (29) become irrelevant for all $1 \leq t \leq 2n - 2$. Accordingly, the relation (26) is used only for $a = -1$ and the other cases are dealt with by (25). A proof of proposition 11 is included in appendix B.

Remark 12. The case (28) is relevant to S_0 only. The operator S_1 is always expressible by (27).

Example 13. For $D_4^{(1)}$, formulae (27)–(29) read

$a = 1$

$$\begin{aligned} S_0 &= \bar{P}^{-1} s_0 K_{-2} Q P \\ S_2 &= \bar{P}^{-1} s_2 s_0 K_{-3} s_0 \bar{P} \\ S_3 &= \bar{P}^{-1} s_3 s_2 s_0 K_{-4} s_0 s_2 \bar{P} \\ S_4 &= \bar{P}^{-1} s_4 s_3 s_2 s_0 K_4 s_0 s_2 s_3 \bar{P} \\ S_2 &= \bar{P}^{-1} s_2 s_4 s_3 s_2 s_0 K_3 s_0 s_2 s_3 s_4 \bar{P} \\ S_0 &= \bar{P}^{-1} s_0 s_2 s_4 s_3 s_2 s_0 K_2 s_0 s_2 s_3 s_4 s_2 \bar{P} \end{aligned}$$

$a = 3$

$$\begin{aligned} S_2 &= P^{-1} s_2 K_2 P \\ S_1 &= P^{-1} s_1 s_2 K_1 s_2 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 s_2 K_{-1} Q s_2 s_1 P \\ S_2 &= \bar{P}^{-1} s_2 s_0 s_1 s_2 K_{-2} s_2 s_1 s_0 \bar{P} \\ S_3 &= \bar{P}^{-1} s_3 s_2 s_0 s_1 s_2 K_{-4} s_2 s_1 s_0 s_2 \bar{P} \\ S_4 &= \bar{P}^{-1} s_4 s_3 s_2 s_0 s_1 s_2 K_4 s_2 s_1 s_0 s_2 s_3 \bar{P} \end{aligned}$$

$a = -4$

$$\begin{aligned} S_4 &= P^{-1} s_4 K_3 P \\ S_2 &= P^{-1} s_2 s_4 K_2 s_4 P \\ S_1 &= P^{-1} s_1 s_2 s_4 K_1 s_4 s_2 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 s_2 s_4 K_{-1} Q s_4 s_2 s_1 P \\ S_2 &= \bar{P}^{-1} s_2 s_0 s_1 s_2 s_4 K_{-2} s_4 s_2 s_1 s_0 \bar{P} \\ S_4 &= \bar{P}^{-1} s_4 s_2 s_0 s_1 s_2 s_4 K_{-3} s_4 s_2 s_1 s_0 s_2 \bar{P} \end{aligned}$$

$a = -2$

$$\begin{aligned} S_2 &= P^{-1} s_2 K_{-3} P \\ S_3 &= P^{-1} s_3 s_2 K_{-4} s_2 P \\ S_4 &= P^{-1} s_4 s_3 s_2 K_4 s_2 s_3 P \\ S_2 &= P^{-1} s_2 s_4 s_3 s_2 K_3 s_2 s_3 s_4 P \\ S_1 &= P^{-1} s_1 s_2 s_4 s_3 s_2 K_1 s_2 s_3 s_4 s_2 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 s_2 s_4 s_3 s_2 K_{-1} Q s_2 s_3 s_4 s_2 s_1 P \end{aligned}$$

$a = 2$

$$\begin{aligned} S_1 &= P^{-1} s_1 K_1 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 K_{-1} Q s_1 P \\ S_2 &= \bar{P}^{-1} s_2 s_0 s_1 K_{-3} s_1 s_0 \bar{P} \\ S_3 &= \bar{P}^{-1} s_3 s_2 s_0 s_1 K_{-4} s_1 s_0 s_2 \bar{P} \\ S_4 &= \bar{P}^{-1} s_4 s_3 s_2 s_0 s_1 K_4 s_1 s_0 s_2 s_3 \bar{P} \\ S_2 &= \bar{P}^{-1} s_2 s_4 s_3 s_2 s_0 s_1 K_3 s_1 s_0 s_2 s_3 s_4 \bar{P} \end{aligned}$$

$a = 4$

$$\begin{aligned} S_3 &= P^{-1} s_3 K_3 P \\ S_2 &= P^{-1} s_2 s_3 K_2 s_3 P \\ S_1 &= P^{-1} s_1 s_2 s_3 K_1 s_3 s_2 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 s_2 s_3 K_{-1} Q s_3 s_2 s_1 P \\ S_2 &= \bar{P}^{-1} s_2 s_0 s_1 s_2 s_3 K_{-2} s_3 s_2 s_1 s_0 \bar{P} \\ S_3 &= \bar{P}^{-1} s_3 s_2 s_0 s_1 s_2 s_3 K_{-3} s_3 s_2 s_1 s_0 s_2 \bar{P} \end{aligned}$$

$a = -3$

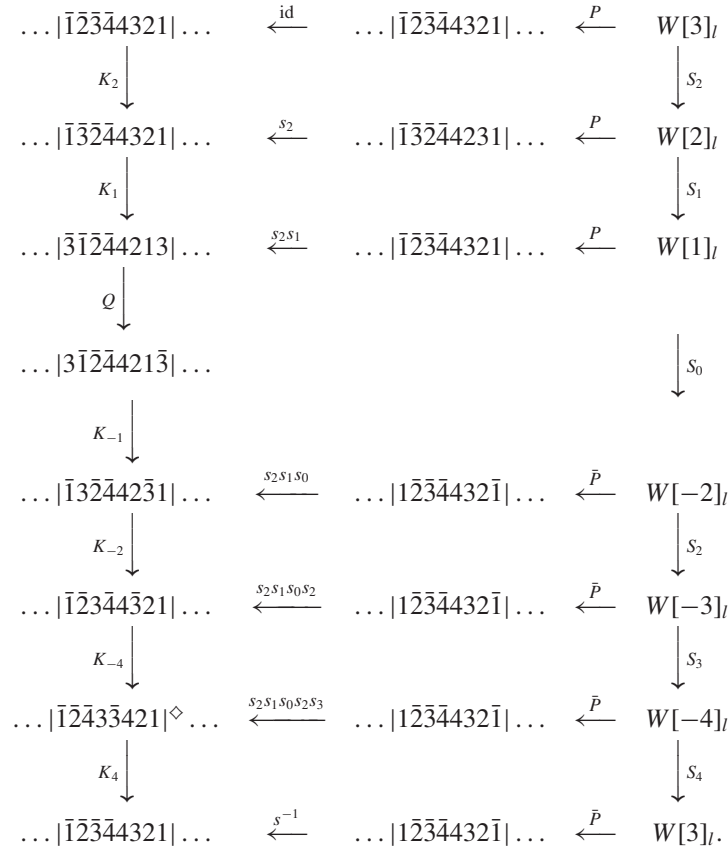
$$\begin{aligned} S_3 &= P^{-1} s_3 K_{-4} P \\ S_4 &= P^{-1} s_4 s_3 K_4 s_3 P \\ S_2 &= P^{-1} s_2 s_4 s_3 K_2 s_3 s_4 P \\ S_1 &= P^{-1} s_1 s_2 s_4 s_3 K_1 s_3 s_4 s_2 P \\ S_0 &= \bar{P}^{-1} s_0 s_1 s_2 s_4 s_3 K_{-1} Q s_3 s_4 s_2 s_1 P \\ S_2 &= \bar{P}^{-1} s_2 s_0 s_1 s_2 s_4 s_3 K_{-2} s_3 s_4 s_2 s_1 s_0 \bar{P} \end{aligned}$$

$a = -1$

$$\begin{aligned} S_1 &= P^{-1} s_1 K_{-2} P \\ S_2 &= P^{-1} s_2 s_1 K_{-3} s_1 P \\ S_3 &= P^{-1} s_3 s_2 s_1 K_{-4} s_1 s_2 P \\ S_4 &= P^{-1} s_4 s_3 s_2 s_1 K_4 s_1 s_2 s_3 P \\ S_2 &= P^{-1} s_2 s_4 s_3 s_2 s_1 K_3 s_1 s_2 s_3 s_4 P \\ S_1 &= P^{-1} s_1 s_2 s_4 s_3 s_2 s_1 K_2 s_1 s_2 s_3 s_4 s_2 P \end{aligned}$$

For instance, in the $a = 1$ case, (24) implies that the first S_0 acts as $W[1]_l \rightarrow W[-2]_l$ whereas the last S_0 acts as $W[2]_l \rightarrow W[-1]_l$. Such a distinction has not been exhibited in the left-hand sides.

Example 14. The commutative diagram corresponding to the $a = 3$ case in example 13.



Here, $\dots | 3\bar{1}\bar{2}\bar{4}421\bar{3} | \dots$ for example indicates that in all the cells the letters from $J = \{\pm 1, \dots, \pm 4\}$ are ordered as

$$3 \dots 3 -1 \dots -1 -2 \dots -2 -4 \dots -4 4 \dots 4 2 \dots 2 1 \dots 1 -3 \dots -3.$$

In the case with (resp. without) the symbol \diamond , $\bar{3}\bar{3}$ may be written as $\bar{3}\bar{3}$ (resp. $\bar{4}\bar{4}$ may be written as $4\bar{4}$) because the cells do not contain these letters simultaneously.

6. Soliton

Let us present the configurations that behave as solitons, namely, some special patterns of local states that undergo only translation under the time evolution $\mathcal{T}_a = T_a$. We concentrate on the basic $D_n^{(1)}$ -automaton but treat the general boundary condition labelled with an arbitrary $a \in J$. The case $a = 1$ was studied in [HKT1]. Solitons in the inhomogeneous $D_n^{(1)}$ -automaton can then be understood along the argument in section 5 of [HKOTY]. See also section III in [HHIKTT]. Note that the operators P and Q become trivial for the basic automaton, hence they can be dropped in definition 3.

Fix $a \in J$ which specifies the boundary condition as in (1). Let $i_1, \dots, i_{2n-2} \in \{0, 1, \dots, n\}$ be any sequence such that $\varepsilon_{i_t}(r_{i_{t-1}} \cdots r_{i_1}(a)) > 0$ for all $1 \leq t \leq 2n-2$ and $r_{i_{2n-2}} \cdots r_{i_1}(a) = r(a)$. (r is defined after (24).) Such a sequence exists for any a and corresponds to a path on the crystal graph (figure 1) going from a to $r(a)$ following the arrows backward. Set $a_t = r_{i_{t-1}} \cdots r_{i_1}(a)$. By the definition $a_{2n-2} = r(a)$ holds. There is no a and

exactly one $-a$ in the array $a_1, a_2, \dots, a_{2n-3}$ for $a \neq \pm 1, \pm n$. For $a = \pm 1, \pm n$, neither a nor $-a$ are contained in the array.

Proposition 15. Consider the following state of the basic $D_n^{(1)}$ -automaton:

$$\dots a \overbrace{a \dots a_1}^{y_1} \overbrace{a_2 \dots a_2}^{y_2} \dots \overbrace{a_{2n-3} \dots a_{2n-3}}^{y_{2n-3}} \overbrace{a_{2n-2} \dots a_{2n-2}}^{y_{2n-2}} a a \dots \quad (30)$$

where \dots on both ends stands for the array of a only, and y_1, \dots, y_{2n-2} are non-negative integers satisfying

$$y_{2n-2} = \begin{cases} 0 & \text{if } a \in \{\pm 1, \pm n\} (\text{i.e., } r(a) \neq a) \\ y_p & \text{otherwise} \end{cases}$$

where p is determined by $a_p = -a$. Under the time evolution $\mathcal{T}_a = T_a$, the underlined pattern is shifted to the right by $y_1 + \dots + y_{2n-2}$ lattice units.

Proof. The case $a = 1$ was proved in [HKT1]. From definition 1, the time evolutions under the other boundary conditions are connected by a similarity transformation as $\mathcal{T}_a = w^{-1}T_1w$, where w is an element of the affine Weyl group. The pattern (30) is obtained by applying w^{-1} to the soliton states [HKT1] for $a = 1$. □

We call the underlined pattern a soliton with amplitude $y_1 + \dots + y_{2n-2}$. Proposition 15 states that the amplitude coincides with the velocity of the solitons. Note that for $a \neq \pm 1, \pm n$, the ‘empty box’ a is regarded also as a part of a soliton. To include them in the right end of solitons is just a convention. Proposition 15 is equally valid if a soliton is defined, instead of (30), as $\dots \overbrace{a \dots a}^{\alpha} a_1, \dots, a_{2n-3} \overbrace{a \dots a}^{\beta} \dots$ for any α and β such that $\alpha + \beta = y_{2n-2}$. If there are more than one solitons, they move independently as described in proposition 15 as long as they are sufficiently separated from each other. As for their scattering theory, see [HKOTY].

Let us explicitly list the possible choices of the array $a_1, a_2, \dots, a_{2n-3}$ for $a \in \{\pm 1, \pm n\}$ and $a_1, a_2, \dots, a_{2n-2}$ for $a \in \{\pm 2, \dots, \pm(n-1)\}$.

$$\begin{aligned} & -2, -3, \dots, -n+1, \left\{ \begin{matrix} n \\ -n \end{matrix} \right\}, n-1, \dots, 3, 2 && \text{for } a = \pm 1 \\ & n-1, \dots, 2, \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right\}, -2, \dots, -n+1 && \text{for } a = \pm n \\ & a-1, a-2, \dots, 2, \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right\}, -2, \dots, -n+1, \left\{ \begin{matrix} n \\ -n \end{matrix} \right\}, n-1, \dots, a+1, a \\ & \text{for } 2 \leq a \leq n-1 \\ & a-1, a-2, \dots, -n+1, \left\{ \begin{matrix} n \\ -n \end{matrix} \right\}, n-1, \dots, 2, \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right\}, -2, \dots, a+1, a \\ & \text{for } -n+1 \leq a \leq -2. \end{aligned}$$

In the first line, the symbol $\left\{ \begin{matrix} n \\ -n \end{matrix} \right\}$ means that both T_1 and T_{-1} admit two arrays corresponding to the choices of n or $-n$. The second line is similar. In the last two lines, each T_a ($a \in \{\pm 2, \dots, \pm(n-1)\}$) admits four arrays.

Example 16. Under the time evolution $T_2 = K_3K_4K_{-4}K_{-3}K_{-1}K_1$ in the basic $D_4^{(1)}$ -automaton, one has (to save space we write $\bar{3}$ to mean -3 etc)

$$\begin{array}{l}
 \dots 221\underline{2\bar{2}\bar{3}}4432222222222222\dots \\
 \xrightarrow{K_1} \dots 222\bar{1}\bar{1}\bar{3}443111222222222222\dots \\
 \xrightarrow{K_{-1}} \dots 22222\bar{3}443\bar{2}\bar{2}122222222222\dots \\
 \xrightarrow{K_{-3}} \dots 22222244\bar{2}331\bar{3}\bar{3}2222222222\dots \\
 \xrightarrow{K_{-4}} \dots 222222444331\bar{3}\bar{3}\bar{4}2222222222\dots \\
 \xrightarrow{K_4} \dots 2222222222331\bar{3}\bar{3}\bar{2}44222222\dots \\
 \xrightarrow{K_3} \dots 222222222221\underline{\bar{2}\bar{2}\bar{3}}44322222\dots
 \end{array}$$

A soliton with amplitude 9 (underlined) is shifted to the right by nine lattice units.

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Appendix A. Crystal structure of B_k for $D_n^{(1)}$

Here we quote the $D_n^{(1)}$ crystal structure of B_k [KKM] in a form adapted to section 5. In this paper we write the Kashiwara operators \tilde{e}_i and \tilde{f}_i ($0 \leq i \leq n$) as e_i and f_i , respectively. Moreover we employ the convention $e_i^q = f_i^{-q}$ for $q < 0$. Let $\vec{x} = (x_1, \dots, x_n, x_{-n}, \dots, x_{-1})$ be an element of the $D_n^{(1)}$ -crystal as described in table 1. For $q \in \mathbb{Z}$, we set $e_i^q(\vec{x}) = \vec{x}'$. We assume that $\vec{x}' \neq 0$ and let $\Delta x_j = x'_j - x_j$ denote the change of the coordinates ($j \in J$). For each $0 \leq i \leq n$, all the Δx_j are zero except the following:

$$i = 0 \quad \begin{cases} \Delta x_{-1} = -\Delta x_2 = (x_2 - x_{-2})_+ - (x_2 - x_{-2} - q)_+ \\ \Delta x_1 = -\Delta x_{-2} = (x_{-2} - x_2)_+ - (x_{-2} - x_2 + q)_+ \end{cases} \tag{31}$$

$$1 \leq i \leq n-1 \quad \begin{cases} \Delta x_i = -\Delta x_{i+1} = (x_{i+1} - x_{-i-1})_+ - (x_{i+1} - x_{-i-1} - q)_+ \\ \Delta x_{-i} = -\Delta x_{-i-1} = (x_{-i-1} - x_{i+1})_+ - (x_{-i-1} - x_{i+1} + q)_+ \end{cases} \tag{32}$$

$$i = n \quad \begin{cases} \Delta x_{n-1} = -\Delta x_{-n} = x_{-n} - (x_{-n} - x_n - q)_+ \\ \Delta x_{-n+1} = -\Delta x_n = x_n - (x_n - x_{-n} + q)_+ \end{cases} \tag{33}$$

Here we have used the symbol $(z)_+ = \max(z, 0)$.

If one associates a node with each element $\vec{x} \in B_k$ and a figure $\vec{x} \xleftarrow{i} \vec{x}'$ with each relation $e_i(\vec{x}) = \vec{x}'$, the result is an oriented graph called a crystal graph, whose arrows carry the colours $0, 1, \dots, n$. For $k = 1$, the crystal graph of B_1 looks like figure 1.

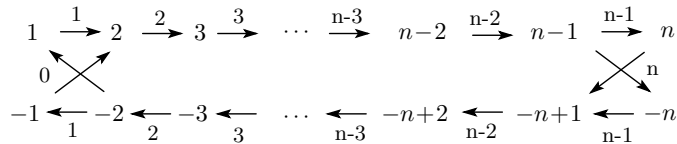


Figure 1. Crystal graph of B_1 .

Here the elements of B_1 are denoted by the letter in J . (See the second paragraph of section 3.)

The functions $\varphi_i, \varepsilon_i : B_k \rightarrow \mathbb{Z}_{\geq 0}$ are given by

$$\varphi_0(\vec{x}) = x_{-1} + (x_{-2} - x_2)_+ \quad \varepsilon_0(\vec{x}) = x_1 + (x_2 - x_{-2})_+ \tag{34}$$

$$\varphi_i(\vec{x}) = x_i + (x_{-i-1} - x_{i+1})_+ \quad \varepsilon_i(\vec{x}) = x_{-i} + (x_{i+1} - x_{-i-1})_+ \quad 1 \leq i \leq n-1 \tag{35}$$

$$\varphi_n(\vec{x}) = x_{n-1} + x_n \quad \varepsilon_n(\vec{x}) = x_{-n+1} + x_{-n}. \tag{36}$$

Note that $(x_j - x_{-j})_+ = x_j$ for $j = \pm n$ due to the constraint $x_n x_{-n} = 0$. For $p \in B_1$, one has the simple description

$$\varepsilon_i(p) = \begin{cases} 1 & \text{if } p' \xrightarrow{i} p \text{ for some } p' \in B_1 \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

The Weyl group operator is defined by $S_i = e_i^{\varepsilon_i - \varphi_i}$. Whenever S_i acts on the states $\xi \in W[a]_l$, it acts as $S_i = e_i^\infty$ since $\varphi_i(\xi) = 0$ holds due to the boundary condition. In such a situation, $\xi' = S_i(\xi)$ can be computed recursively as

$$\xi = (\dots \vec{\xi}_g, \vec{\xi}_{g+1}, \dots) \quad \xi' = (\dots \vec{\xi}'_g, \vec{\xi}'_{g+1}, \dots) \tag{38}$$

$$\vec{\xi}'_g = e_i^{(\varepsilon_i(\vec{\xi}_g) - m_g)_+}(\vec{\xi}_g) \quad m_{g+1} = (m_g - \varepsilon_i(\vec{\xi}_g))_+ + \varphi_i(\vec{\xi}_g) \tag{39}$$

$$\vec{\xi}'_g = e_i^{\varepsilon_i(\vec{\xi}_g)}(\vec{\xi}_g) \quad m_g = 0 \quad \text{for } g \ll -1. \tag{40}$$

The $m_{g+1} \in \mathbb{Z}_{\geq 0}$ determined as above is equal to $\varphi_i((\dots, \vec{\xi}_{g-1}, \vec{\xi}_g))$.

Appendix B. Proof of proposition 11

Here we exhibit the dependence of the operators K_j and Q on the boundary condition $a \in J$ as $K_j^{(a)}$ and $Q^{(a)}$. By definition it is easy to see

Lemma 17. For any $0 \leq i \leq n$ and $a, j \in J$, the following is valid:

$$s_i K_j^{(a)} = K_{r_i(j)}^{(r_i(a))} s_i \tag{41}$$

$$s_i Q^{(a)} = Q^{(r_i(a))} s_i. \tag{42}$$

We denote by $S_i|_{W[a]_l}$ the restriction of S_i on $W[a]_l$.

Lemma 18.

$$S_0|_{W[1]_l} = \bar{P}^{-1} s_0 K_{-2}^{(1)} Q^{(1)} P \quad S_0|_{W[2]_l} = \bar{P}^{-1} s_0 K_{-1}^{(2)} \bar{P} \tag{43}$$

$$S_1|_{W[2]_l} = P^{-1} s_1 K_1^{(2)} P \quad S_1|_{W[-1]_l} = P^{-1} s_1 K_{-2}^{(-1)} P \tag{44}$$

$$S_i|_{W[c]_l} = P^{-1} s_i K_{r_i(c)}^{(c)} P = \bar{P}^{-1} s_i K_{r_i(c)}^{(c)} \bar{P} \quad (i \neq 0, 1) \tag{45}$$

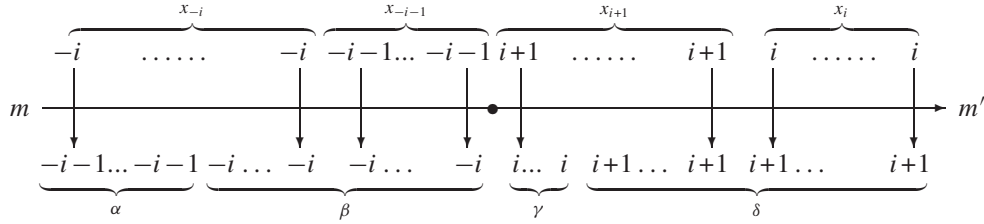
where in (45), c is any element of $J = B_1$ satisfying $\varepsilon_i(c) > 0$. (For any $i \neq 0, 1$, there are two such c , which are contained in the crystal graph as $r_i(c) \xrightarrow{i} c$.)

Proof. Let us show $S_i|_{W[i+1]_l} = P^{-1} s_i K_i^{(i+1)} P$ for $2 \leq i \leq n-1$, which is a part of the assertion (45). The other cases can be verified similarly. Obviously, the map

$s_i K_i^{(i+1)} : \widetilde{W}[i+1]_l \rightarrow \widetilde{W}[i]_l$ only changes the letters $\pm i, \pm(i+1)$ in each cell. So setting $\vec{\xi}_g = (x_1, \dots, x_{-1}) \in B_{l_g}$, we only consider their array within a cell:

$$\overbrace{-i, \dots, -i}^{x_{-i}} \overbrace{-i-1, \dots, -i-1}^{x_{-i-1}} \overbrace{i+1, \dots, i+1}^{x_{i+1}} \overbrace{i, \dots, i}^{x_i}$$

dropping the other letters. When $K_i^{(i+1)}$ is applied on this array according to the vertex diagram rule (9), it is transformed as



where we have assumed that the number attached to the horizontal line as in (9) is m for the cell in question. Here $\alpha, \beta, \gamma, \delta$ and m' are given by

$$\alpha = \min(x_{-i}, m) \quad \beta = x_{-i-1} + (x_{-i} - m)_+ \tag{46}$$

$$\gamma = \min(x_{i+1}, \tilde{m}) \quad \delta = x_i + (x_{i+1} - \tilde{m})_+ \tag{47}$$

$$m' = x_i + (\tilde{m} - x_{i+1})_+ \quad \tilde{m} = x_{-i-1} + (m - x_{-i})_+ \tag{48}$$

where \tilde{m} is the number attached to the horizontal line at the position marked with the symbol \bullet . Thus under $P^{-1} s_i K_i^{(i+1)} P$, the element $\vec{\xi}_g$ is transformed into $\vec{\xi}'_g = (x'_1, \dots, x'_{-1}) \in B_{l_g}$ with non-trivial changes given by $x'_{-i} = \alpha, x'_{-i-1} = \beta, x'_{i+1} = \gamma$ and $x'_i = \delta$. We are to show that this coincides with the rule (38)–(40) for S_i . In view of (32), it is equivalent to checking

$$\begin{aligned} \delta - x_i &= x_{i+1} - \gamma = (x_{i+1} - x_{-i-1})_+ - (x_{i+1} - x_{-i-1} - q)_+ \\ \alpha - x_{-i} &= x_{-i-1} - \beta = (x_{-i-1} - x_{i+1})_+ - (x_{-i-1} - x_{i+1} + q)_+ \\ m' &= (m - x_{-i} - (x_{i+1} - x_{-i-1})_+) + x_i + (x_{-i-1} - x_{i+1})_+ \end{aligned}$$

where $q = (x_{-i} + (x_{i+1} - x_{-i-1})_+ - m)_+$. These identities can be proved straightforwardly. \square

Example 19. For $n = 4$, the relations (45) read

$$S_2|_{W[3]_l} = P^{-1} s_2 K_2^{(3)} P = \bar{P}^{-1} s_2 K_2^{(3)} \bar{P} \tag{49}$$

$$S_2|_{W[-2]_l} = P^{-1} s_2 K_{-3}^{(-2)} P = \bar{P}^{-1} s_2 K_{-3}^{(-2)} \bar{P} \tag{50}$$

$$S_3|_{W[4]_l} = P^{-1} s_3 K_3^{(4)} P = \bar{P}^{-1} s_3 K_3^{(4)} \bar{P} \tag{51}$$

$$S_3|_{W[-3]_l} = P^{-1} s_3 K_{-4}^{(-3)} P = \bar{P}^{-1} s_3 K_{-4}^{(-3)} \bar{P} \tag{52}$$

$$S_4|_{W[-4]_l} = P^{-1} s_4 K_3^{(-4)} P = \bar{P}^{-1} s_4 K_3^{(-4)} \bar{P} \tag{53}$$

$$S_4|_{W[-3]_l} = P^{-1} s_4 K_4^{(-3)} P = \bar{P}^{-1} s_4 K_4^{(-3)} \bar{P}. \tag{54}$$

Proof of proposition 11. All the relations (27)–(29) are reduced to lemma 18 by means of lemma 17. In fact consider the case $i_t \neq 0, 1$. Then from remark 12, the identities to be shown are either (27) or (29). With the aid of lemma 17, they are reduced to

$$S_{i_t} |W[r_{i_{t-1} \dots r_{i_1}(a)}]_l = P^{-1} s_{i_t} K_{r_{i_{t-1} \dots r_{i_1}(j_t)}^{(r_{i_{t-1} \dots r_{i_1}(a)})} P \quad \text{or} \quad \bar{P}^{-1} s_{i_t} K_{r_{i_{t-1} \dots r_{i_1}(j_t)}^{(r_{i_{t-1} \dots r_{i_1}(a)})} \bar{P} \quad (55)$$

where on the left-hand side, we have exhibited the fact (24) explicitly. Substitute (23) into this and set $r_{i_{t-1}} \dots r_{i_1}(a) = c$ and $i_t = i$. The result coincides with (45). Moreover, the property (21) tells that $\varepsilon_i(c) > 0$. Thus (55) is reduced to lemma 18. The remaining case $i_t = 0, 1$ can be shown similarly by means of (43) and (44). This completes the proof of proposition 11. \square

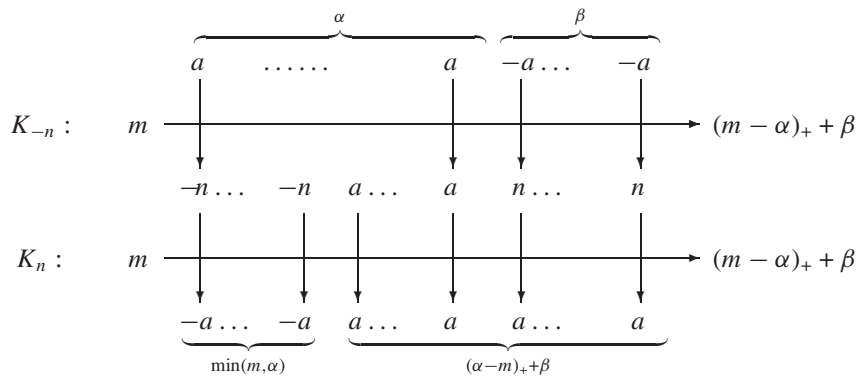
Remark 20. By substituting (45) into $S_{n-1}S_n = S_nS_{n-1}$ and applying lemma 17, one can show $K_n^{(a)}K_{-n}^{(a)} = K_{-n}^{(a)}K_n^{(a)}$ for $a \neq \pm n$. Similarly, $S_0S_1 = S_1S_0$ leads to $Q^{(1)}K_{-1}^{(a)}Q^{(a)}K_1^{(a)} = K_1^{(a)}Q^{(a)}K_{-1}^{(a)}Q^{(1)}$ for $a \neq \pm 1$.

Finally we include a lemma concerning the operator K_{-a} appearing in the $A_{2n-3}^{(2)}$ -automaton and the vertex diagram (14).

Lemma 21. If $x_n = x_{-n} = 0$ for all the local states $\vec{x} = (x_1, \dots, x_{-1})$ in the $D_n^{(1)}$ -automaton, the composition K_nK_{-n} appearing in T_a ($a \neq \pm n$) is equal to K_{-a} defined around (14).

Proof. From the remark made after example 4, the K_nK_{-n} always acts on the cells of the

form $\dots \overbrace{a \dots a}^\alpha \overbrace{-a \dots -a}^\beta \dots$, where the \dots parts on both ends do not contain the letters $\pm a, \pm n$ and are kept unchanged. Therefore by dropping them, the action of K_nK_{-n} on a cell can be computed with the following diagram:



One finds that the letters $\pm n$ are indeed confined within the intermediate state between K_n and K_{-n} . Moreover, the composition K_nK_{-n} induces the same transformation as K_{-a} defined by (14). \square

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